International Journal of Theoretical Physics, Vol. 35, No. 9, 1996

Quantum Spacetime

A. R. Marlow[1]

Received November 6, 1995

After a review and reformulation of previous results, further developments in the construction and interpretation of a quantum-theoretic model of general relativistic spacetime are presented. A theorem is proved that clarifies the nature of the boundaries separating the four-dimensional facets of the resulting piecewise linear model, and a simple example of such a spacetime is detailed. The precise way in which any such model unifies the essential features of both quantum theory and general relativity is discussed.

In a previous paper (Marlow, 1995, hereafter referred to as I) a construction of general relativistic spacetime as a piecewise linear manifold within the structure of quantum observables was presented; here, after a review and reformulation of that construction and its motivation, some further results are established for the model and its interpretation.

1. BASIC FORMULATION

The only input needed for the construction of quantum spacetime is an ordering parameter t for states in quantum processes—to begin we will assume that t runs through a finite collection of proper time instants for some observer, but later other parameters may be equally useful. (Initially, the finiteness requirement simply ensures that the construction is well defined, and allows us to avoid the complicating distraction of discussing limiting procedures on a first formulation; as will become evident, however, finiteness, or at least countability, seems to be inherent in the actual physical processes

[1] Department of Physics, Loyola University, New Orleans, Louisiana 70118; e-mail: marlow@beta.loyno.edu.

as well.) Then, on noting that a product of representations of the identity operator on a Hilbert space is still the identity operator

$$I \equiv \prod_t I_t$$

and allowing each I_t to stand for a Dirac-style expansion of the identity in terms of a complete orthogonal set of one-dimensional projections

$$I_t = \sum_{n_t} |n_t\rangle\langle n_t|$$

where each index n_t runs through the integer labels for a complete orthonormal basis of the Hilbert space, we have

$$I = \prod_t \left(\sum_{n_t} |n_t\rangle\langle n_t| \right)$$

as a possible expansion of the identity. Then a simple reversal of the order of product and summation yields a Feynman-style expansion of the identity as a *sum over quantum processes*:

$$I = \sum_n \left(\prod_t |n(t)\rangle\langle n(t)| \right) \tag{1}$$

where now each n in the summation represents an integer-valued function $t \to n(t)$ of the ordering parameter t, and each product

$$\prod_t |n(t)\rangle\langle n(t)|$$

specifies a possible quantum process with quantum states represented by unit vectors $|n(t)\rangle$ as intermediate stages. Sequences of quantum state projections such as

$$|n(t)\rangle\langle n(t)| \cdots |n(t')\rangle\langle n(t')| \cdots |n(t'')\rangle\langle n(t'')|$$

above are usually referred to as *quantum histories* in the literature on the consistent history interpretation of quantum theory [for a review article, see Saunders (1993)]; we will use the terms *processes* and *histories* interchangeably.

We have derived Feynman's basic expansion of the identity as a sum over alternative quantum histories (1) in the above way, first to show that it can be done without any reference to a preexisting four-dimensional space-time, and then to set the stage for confronting the basic question remaining unanswered in 20th-century physics: How do quantum processes, originally represented purely as sequences of projection operators on an infinite-dimensional Hilbert space, acquire four-dimensional spacetime labels, or alternatively, how do the four dimensions of spacetime arise from pure quantum theory as an apparent background for quantum processes? Obviously, in raising this question we are implicitly rejecting the usual procedure of simply inserting the classical dimensions by hand as a sort of background "ether" for quantum theory. As is clear from the previous paper, we believe that the four-dimensional appearance of the observed universe naturally arises from the structure of quantum theory itself, and hence we assume this latter theory alone as the best fundamental account of physical processes yet discovered, without any need for supplemental four-dimensional assumptions.

To continue the review of how this comes about, we note that every two distinct state operators $|n(t_a)\rangle\langle n(t_a)|$ and $|n(t_b)\rangle\langle n(t_b)|$ in a single quantum process of the expansion (1) uniquely specify the four-space $\mathcal{S}^4_{ab}$ of self-adjoint operators on the complex two-dimensional Hilbert subspace $\mathcal{H}^2_{ab}$ spanned by $|n(t_a)\rangle$ and $|n(t_b)\rangle$. A standard operator basis for $\mathcal{S}^4_{ab}$ consists of Pauli operators on $\mathcal{H}^2_{ab}$, e.g.,

$$\begin{aligned}
\sigma_{ab0} &= |n(t_a)\rangle\langle n(t_a)| + |n_\perp(t_b)\rangle\langle n_\perp(t_b)| \\
\sigma_{ab1} &= |n(t_a)\rangle\langle n_\perp(t_b)| + |n_\perp(t_b)\rangle\langle n(t_a)| \\
\sigma_{ab2} &= i(|n_\perp(t_b)\rangle\langle n(t_a)| - |n(t_a)\rangle\langle n_\perp(t_b)|) \\
\sigma_{ab3} &= |n(t_a)\rangle\langle n(t_a)| - |n_\perp(t_b)\rangle\langle n_\perp(t_b)|
\end{aligned} \tag{2}$$

where $|n_\perp(t_b)\rangle$ is the uniquely defined (up to a complex phase) unit vector orthogonal to $|n(t_a)\rangle$ in $\mathcal{H}^2_{ab}$.

It is the space $\mathcal{S}^4_{ab}$ of self-adjoint operators on $\mathcal{H}^2_{ab}$ that we take as the definition of the spacetime "between" or "connecting" the states defined by $|n(t_a)\rangle$ and $|n(t_b)\rangle$. More technically, $\mathcal{S}^4_{ab}$ will define the tangent four-plane to the region of our piecewise linear general relativistic manifold containing the quantum events specified by the pair of states. In terms of the basis $\sigma_{ab\mu}$, $\mu = 0, 1, 2, 3$, defined above, a general pair of elements of such a spacetime is (using the standard summation convention)

$$x = x^\mu \sigma_{ab\mu}, \qquad y = y^\nu \sigma_{ab\nu}$$

while the usual Einstein metric is specified, in terms of the trace function, as

$$g(x, y) \equiv 1/2 \operatorname{tr}(x^c y) \tag{3}$$

where

$$x^c \equiv x^0 \sigma_{ab0} - x^i \sigma_{abi}$$

defines the operation of *space inversion* (reversal of sign of the trace zero elements of $\mathcal{S}^4_{ab}$). Baylis *et al.* (1992) present a persuasive case for the Pauli operator structure (including the full Clifford algebra of all operators on $\mathcal{H}^2_{ab}$) as the natural structure for special relativistic physics; to the author's knowledge, the present work and I are the first instances where it is argued that the structure is equally natural for general relativistic physics.

To see precisely how $\mathcal{S}^4_{ab}$ defines the spacetime between $|n(t_a)\rangle$ and $|n(t_b)\rangle$, suppose we perform a general Feynman *sum-over-processes* analysis of the quantum amplitude $\langle n(t_b)|n(t_a)\rangle$, that is, we insert an expansion of the identity such as (1) above to get

$$\langle n(t_b)|n(t_a)\rangle = \langle n(t_b)|\sum_n \left(\prod_t |n(t)\rangle\langle n(t)|\right)|n(t_a)\rangle \tag{4}$$

Then, while very general processes appear in the abstract expansion (1), the *only* processes that make a nonzero contribution to (3) are those that contain no operator components orthogonal to either $|n(t_b)\rangle$ or $|n(t_a)\rangle$ [Proof: From the construction of (1) only copies I_t of the identity operator separate $|n(t_b)\rangle$ and $|n(t_a)\rangle$ from any intervening operators, and so any operators in (3) orthogonal to either of the given vectors get projected out and make no contribution to the analysis.] Hence, the only part of any vector $|n(t)\rangle$ in (3) that makes a nonzero contribution to the analysis can be written as

$$\sigma_{ab0}|n(t)\rangle = C_a(t)|n(t_a)\rangle + C_b(t)|n(t_b)\rangle$$

$$C_a(t) \neq 0, \qquad C_b(t) \neq 0$$

and so a complete sum over all processes making a nonzero contribution in (3) can be expressed as

$$\begin{aligned}\langle n(t_b)|n(t_a)\rangle &= \langle n(t_b)|\sum_n \left[\prod_t \sigma_{ab0}|n(t)\rangle\langle n(t)|\sigma_{ab0}\right]|n(t_a)\rangle \\ &= \langle n(t_b)|\sum_n \left[\prod_t \lambda_{ab}(t)\right]|n(t_a)\rangle \end{aligned} \tag{5}$$

We now take over from I the easily checked fact that the compression

$$\lambda_{ab}(t) = \sigma_{ab0}|n(t)\rangle\langle n(t)|\sigma_{ab0}$$

of arbitrary projections such as $|n(t)\rangle\langle n(t)|$ into $\mathcal{S}^4_{ab}$ yields only *lightlike* elements, i.e.,

$$\lambda_{ab}(t) = \tfrac{1}{2}p(t)[\sigma_{ab0} + \sigma_{ab}(t)]$$

where

$$p(t) \equiv \langle n(t) | \sigma_{ab0} | n(t) \rangle \neq 0$$

and $\sigma_{ab}(t)$ is the spacelike (that is, zero-trace) element of $\mathcal{S}^4_{ab}$ given by

$$\begin{aligned}\sigma_{ab}(t) = {} & \sin\theta(t)\cos\phi(t)\,\sigma_{ab1} \\ & + \sin\theta(t)\sin\phi(t)\,\sigma_{ab2} + \cos\theta(t)\,\sigma_{ab3}\end{aligned} \tag{6}$$

with

$$\begin{aligned}\cos\theta(t) &= p(t)^{-1}[|\langle n(t_a)|n(t)\rangle|^2 - |\langle n_\perp(t_b)|n(t)\rangle|^2] \\ \sin\theta(t)\cos\phi(t) &= p(t)^{-1}[\langle n_\perp(t_b)|n(t)\rangle\langle n(t)|n(t_a)\rangle \\ &\quad + \langle n(t_a)|n(t)\rangle\langle n(t)|n_\perp(t_b)\rangle] \\ \sin\theta(t)\sin\phi(t) &= ip(t)^{-1}[\langle n_\perp(t_b)|n(t)\rangle\langle n(t)|n(t_a)\rangle \\ &\quad - \langle n(t_a)|n(t)\rangle\langle n(t)|n_\perp(t_b)\rangle]\end{aligned}$$

To summarize, a Feynman sum-over-processes analysis in terms of possible quantum processes connecting any two distinct quantum states naturally restricts to only lightlike processes in a four-space of observables. As in I, now we argue that the piecing together of such four-spaces of quantum observables, $\mathcal{S}^4_{ab}$, $\mathcal{S}^4_{bc}$, etc., very much as in the construction of one of Buckminster Fuller's geodesic domes, can give a viable theory of the quantum origin of general relativistic spacetime. The obvious advantage of such a theory is that it removes any potential conflict between general relativity and quantum theory by actually embedding the structure of spacetime in the observable structure of quantum theory—the two theories will no longer be in rivalry, but one will have its complete existence within the other, where it can be interpreted accordingly. The basic difference between standard general relativistic manifolds and the type proposed here, aside from the fact that our manifolds are realized in terms of quantum observables, is that the quantum manifolds are only piecewise smooth, with nondifferentiability occurring at the boundaries separating the different four-planes; we note that such manifolds are also piecewise linear, with all curvature concentrated at the same boundaries.

It is this concentration of curvature or bending at the boundaries separating the quantum four-spaces that, in paper I, we associated with the presence of matter. In doing this, we emphasize the important distinction that must be made in any Feynman-type analysis between *actualized states* (i.e., those that are actually prepared or in some way irreversibly recorded) and those that are merely *potential*. Potential states or processes provide alternatives that must be summed over in analyzing and computing quantum amplitudes simply because no one of them has been given any more status in reality than the others, while actualized states rule out other competing alternatives and are not summed over in computing quantum amplitudes. It is these latter actualized states that we associate with matter and the boundaries defining distinct four-planes, while the merely potential states get tailored by the actual states to fit into and define the vacuum light cones connecting the actual states. Since the actualized states of matter imply a permanent record, the situation was summed up in I by the aphorism "Matter is memory."

2. FURTHER RESULTS

In presenting the above structure in I as a candidate for the microscopic quantum structure of general relativistic spacetime, no specific example of such a manifold was presented. Here we remedy that, first proving a result concerning the possible bounding intersections that can be present in a piecewise linear manifold formed from four-planes $\mathcal{S}^4$ of self-adjoint operators on various two-dimensional subspaces $\mathcal{H}^2$ of Hilbert space.

Theorem. Given two distinct four-spaces $\mathcal{S}_1^4$ and $\mathcal{S}_2^4$ consisting of the self-adjoint operators on two-dimensional subspaces $\mathcal{H}_1^2$ and $\mathcal{H}_2^2$ of a complex Hilbert space, if the intersection $\mathcal{S}_1^4 \cap \mathcal{S}_2^4$ contains nonzero operators, then $\mathcal{S}_1^4 \cap \mathcal{S}_2^4$ consists of all real multiples of some projection $|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a unit vector in $\mathcal{H}_1^2 \cap \mathcal{H}_2^2$. [Stated in the obvious relativistic terminology, this means that if two distinct operator spacetimes have a nontrivial intersection, that intersection can consist only of a single lightlike ray—neither spacelike nor timelike rays may occur in the intersection.]

Proof. The theorem is almost obvious. To make it completely so, classify the operators in $\mathcal{S}_1^4$ and $\mathcal{S}_2^4$ according to the dimension of their range spaces (i.e., according to *rank*), and note that if any operators with two-dimensional range were in the intersection, then we must have $\mathcal{H}_1^2 = \mathcal{H}_2^2$, and so $\mathcal{S}_1^4 = \mathcal{S}_2^4$, contradicting the premise of distinct operator spaces. By inspection, this rules out the operators constructed in (2) above, and the explicit exclusion of the trivial rank-0 case (the zero operator) leaves only the rank-1 operators. Since these are all multiples of projections such as $|\psi\rangle\langle\psi|$, and since no more than one such projection can be shared in common if $\mathcal{S}_1^4 \neq \mathcal{S}_2^4$, the theorem follows.

Finally, writing the identities

$$
\begin{aligned}
|\psi\rangle\langle\psi| &= \tfrac{1}{2}[|\psi\rangle\langle\psi| + |\psi_1^\perp\rangle\langle\psi_1^\perp|] + \tfrac{1}{2}[|\psi\rangle\langle\psi| - |\psi_1^\perp\rangle\langle\psi_1^\perp|] \\
&= \tfrac{1}{2}[\sigma_{10} + \sigma_{1\psi}]
\end{aligned}
$$

$$= \tfrac{1}{2}[\,|\psi\rangle\langle\psi| + |\psi_2^\perp\rangle\langle\psi_2^\perp|\,] + \tfrac{1}{2}[\,|\psi\rangle\langle\psi| - |\psi_2^\perp\rangle\langle\psi_2^\perp|\,]$$
$$= \tfrac{1}{2}[\sigma_{20} + \sigma_{2\psi}]$$

makes clear the lightlike structure of the rank-1 operator $|\psi\rangle\langle\psi|$ in both spaces $\mathcal{S}_1^4$ and $\mathcal{S}_2^4$, as claimed, where $|\psi_1^\perp\rangle$ and $|\psi_2^\perp\rangle$ are unit vectors orthogonal to $|\psi\rangle$ in $\mathcal{H}_1^2$ and $\mathcal{H}_2^2$, respectively.

Thus the only "bends" or "creases" in a piecewise linear four-manifold of operators such as we propose here must lie solely along lightlike directions defined by pure quantum states. Assuming now that any *actualized* quantum history will consist of a discrete sequence of distinct states ordered by instants of proper time for some observer, e.g.,

$$\cdots\ |\psi(t_1)\rangle\langle\psi(t_1)|\ \cdots\ |\psi(t_2)\rangle\langle\psi(t_2)|\ \cdots\ |\psi(t_3)\rangle\langle\psi(t_3)|\ \cdots$$

it is straightforward to see how each such history naturally generates its own piecewise linear spacetime manifold $\mathcal{S}^4$ of quantum observables: since each pair of states compresses all potential quantum processes connecting the states into the operator spacetime defined by the pair, we can symbolically represent the manifold as the sequence of tangent four-planes intersecting in the lightlike rays spanned by the elements

$$\lambda(t_i) = |\psi(t_i)\rangle\langle\psi(t_i)|$$
$$= \tfrac{1}{2}[\sigma_{i-1,i,0} + \sigma_{i-1,i}(t_i)]$$
$$= \tfrac{1}{2}[\sigma_{i,i+1,0} + \sigma_{i,i+1}(t_i)]$$

i.e., for $\Lambda_i \equiv \{r\lambda(t_i)\colon r \in \mathbb{R}\}$,

$$\cdots \cap \mathcal{S}_{01}^4 \cap \Lambda_1 \cap \mathcal{S}_{12}^4 \cap \Lambda_2 \cap \mathcal{S}_{23}^4 \cap \Lambda_3 \cap \mathcal{S}_{34}^4 \cap \cdots$$

We can parametrize such an operator-valued manifold $\mathcal{S}^4$ in an obvious way as the four-parameter family $x(t, r, \theta, \phi)$ of self-adjoint operators defined by

$$x(t, r, \theta, \phi) = t\sigma_{i,i+1,0} + r\sigma_{i,i+1}(\theta, \phi) \tag{7}$$

for

$$t_i < t < t_{i+1}, \qquad r, \theta, \phi \text{ arbitrary}$$

where $\sigma_{i,i+1,0}$ and $\sigma_{i,i+1}(\theta, \phi)$ are unit timelike and spacelike components as in (2) and (6) (with θ and ϕ independent parameters now with no dependence on t), and at each t_i there is the dimensional singularity defined by the state $|\psi(t_i)\rangle\langle\psi(t_i)| = \lambda(t_i)$, where the dimensionality of the manifold reduces from four to one. As noted before, we interpret these lightlike one-dimensional "boundaries" as indicating the actual presence of matter (an irreversibly recorded state) as opposed to the potentiality for matter, represented by the four-dimensional vacuum spacetimes connecting the singularities.

To give an interpretation of the potential trajectories in the vacuum spacetime we can use either the Schrödinger picture (evolving states) or the Heisenberg picture (evolving observables), and, as a general principle, we find that all of the standard structures of quantum theory and quantum field theory project onto the piecewise linear four-manifold M defined by a sequence of states $\lambda_i = |\psi_i\rangle\langle\psi_i|$. Specifically, over each linear piece $\mathcal{S}^4_{i,i+1}$ of M, define the projection $\Pi_{i,i+1}$ on the space A of all self-adjoint operators α on the Hilbert space of quantum theory by

$$\Pi_{i,i+1}\alpha = \sigma_{i,i+1,0}\alpha\sigma_{i,i+1,0}$$

This provides a general bundle structure for self-adjoint operators over $\mathcal{S}^4_{i,i+1}$, with the fiber over any $x \in \mathcal{S}^4_{i,i+1}$ defined as the space of operators α such that $\Pi_{i,i+1}\alpha = x$. We posit this basic structure as the general framework for all the successful gauge theories for particle fields over Minkowski space, and we can now begin to see how such theories might be extended to our piecewise linear general relativistic manifold.

Thus, from the Feynman sum-over-processes formulation of quantum theory we are inevitably led to replace the locally Lorentzian manifold of general relativity at a fundamental quantum level with the almost everywhere linear Minkowskian geometry of Pauli operators on subspaces of Hilbert space as defined above. Over the Minkowski linear faces of the resulting geometric complex, standard relativistic quantum theory, including quantum field theory, is assumed to hold, where we propose the usual $p_\mu = i\partial_\mu$ ($\hbar = 1$) identification of derivatives along the σ_μ axes with energy-momentum observables. Thus, quantum uncertainty is built into the foundations of our general relativistic model from the start, since information is extracted from the model only via the usual quantum expectation value formula $\overline{\alpha} = \langle\psi|\alpha|\psi\rangle$, with uncertainty intrinsically included by $\Delta\alpha = [\overline{\alpha^2} - \overline{\alpha}^2]^{1/2}$. Of course, as the correspondence principle requires, at a macroscopic level our piecewise linear quantum manifold would mimic the locally Lorentz manifold of standard general relativity, except at the boundary singularities, which we interpret as nonvacuum states. It is these dimensional singularities as boundaries that give our model its general relativistic character, since they make our base manifold only piecewise linear, and hence allow the nonlinear behavior characteristic of topologically more interesting manifolds. An open question remains as to the precise formulation of physics at such boundaries—their one-dimensionality might suggest strings of some sort, though it would be premature to make such an identification at this point—but this seems no more of a problem than the interpretation of singularities in any general relativistic model, and at least in the construction presented here, we can begin to see how such dimensional singularities arise naturally from quantum theory.

REFERENCES

Baylis, W., Hushilt, J., and Wei, J. (1992). *American Journal of Physics*, **60**(9), 788–797.
Marlow, A. R. (1995). *International Journal of Theoretical Physics*, **34**(8), 1559–1565.
Saunders, S. (1993). *Foundations of Physics*, **23**(12), 1553–1585.